wing is 3×6. The results for delta wings of $AR = 1.07$ and 1.865 with leading-edge separation are presented in Fig. 3. The mesh on the half-wing is 5×5 and 6×6, respectively. Experimental data in both figures are obtained from Ref. 3. It can be seen that the results of the present method agree well with the experiment.

The calculated wing loads from the reduced linear basic equation are higher than the present results. This conclusion was also obtained by the integral method of Ref. 1.

From these examples, it is found that the nonlinear terms of the basic equation are as important for a moderate Mach number as a higher Mach number. And for these double sources of nonlinearity problem, the localized linearization method is a better approach in respect to convergence of computation. Besides, this method is much more economical as the number of panels used is considerably less than the integral method that involves additional three-dimensional space discretization.

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Application of the Boundary Element Method to the Thin Airfoil Theory

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I. Introduction

IN the conventional theory of the thin airfoils (see Ref. 1), the following two symplifying hypotheses are assumed: 1) the boundary condition is linearized and 2) this condition is transferred on the chord of the airfoil. In the theory presented here these hypotheses are given up (hence the exact condition is used on the airfoil, i.e., the natural setting).

In the literature there exists the paper by Hess and Smith[2] where the boundary element method (BEM) is used for nonlifting three-dimensional bodies in incompressible fluids, but these authors use the so-called indirect method[3] assimilating the body surface with a source distribution. The method used here differs from the Hess-Smith method in that the method used here is direct and does not assimilate the body with a source distribution. As it is known,[4] in this type of problem, the direct methods give better results than the indirect methods.

The idea underlying BEM is to use the fundamental solutions of the equation of motion in order to reduce the boundary-value problem to an integral equation on the body boundary and then to solve this equation by discretization. In the present paper, we prefer to consider the fundamental quantities p and v and not the velocity potential as conventional because we intend to obtain and equation in p. The aerodynamics of interest are the pressure values on the wings and not the values of the potential.

II. Equations of Motion

We want to determine the perturbation produced in a uniform subsonic stream of velocity U_∞, pressure p_∞, and density ρ_∞ by an airfoil C. We use the reference frame x_1Oy_1 with the Ox_1 axis in the direction of the unperturbed stream and O at the leading edge. We introduce the dimensionless variables X, Y defined by relation $(x_1,y_1) = L_0(X,Y)$, L_0 being the length of the airfoil chord. Denoting V_1 the total velocity and P_1 the total pressure, we have

$$V_1 = U_\infty(1+V), \qquad P_1 = p_\infty + \rho_\infty U_\infty^2 P \tag{1}$$

V and P being the dimensionless perturbation velocity and pressure, respectively, determined by system[5]

$$M_\infty^2 \partial P/\partial X + \text{Div } V = 0, \qquad \partial V/\partial X + \text{ Grad } P = 0 \tag{2}$$

and boundary condition

$$(1+V)\cdot N = 0 \text{ on } C \tag{3}$$

and damping condition

$$\lim (P, V) = 0 \tag{4}$$

We denote M_∞ (<1) the Mach number in the free flow and N the inner normal to C. With $V = (U, V)$ from Eq. (2) we deduce $U = -P$ and

$$\beta^2 \partial P/\partial X - \partial V/\partial Y = 0 \qquad \partial V/\partial X + \partial P/\partial Y = 0 \tag{5}$$

where $\beta = \sqrt{1 - M_\infty^2}$

Performing the change of variable X, $Y \to x,y$

$$x = X, \qquad y = \beta Y \tag{6}$$

and the change of functions $P, V \to p, v$

$$p = \beta P, \qquad v = V \tag{7}$$

the system of Eq. (5) becomes

$$\partial p/\partial x - \partial v/\partial y = 0, \qquad \partial v/\partial x + \partial p/\partial y = 0 \tag{8}$$

and the boundary condition of Eq. (3) reads

$$pn_1 - vn_2 = \beta n_1 \tag{9}$$

and the damping condition becomes

$$\lim_\infty (p,v) = 0 \tag{10}$$

In Eq. (9) $\boldsymbol{n} = (n_1, n_2)$ is the unit vector of the interior normal to the boundary C in the new variables.

III. Integral Equation

Using the Fourier transform method, it is shown that the solution of system

$$\partial p^*/\partial x - \partial v^*/\partial y = \delta(\boldsymbol{x} - \xi), \quad \partial v^*/\partial x + \partial p^*/\partial y = 0 \tag{11}$$

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where δ is the Dirac function $[\delta(z-\zeta)=\delta(x-\zeta)\delta(y-\eta)]$ is

$$p^*(x-\xi)=\frac{1}{2\pi}\frac{x-\xi}{(x-\xi)^2+(y-\eta)^2},\qquad v^*(x-\xi)=-\frac{1}{2\pi}$$

$$\frac{y-\eta}{(x-\xi)^2+(y-\eta)^2} \tag{12}$$

This is a fundamental solution of the system of Eqs. (8) and (9). The method is known from Ref. 6.

Multiplying the first part of Eq. (8) by $-p^*$, integrating over the domain D filled with fluid and using the flux – divergence formula, and taking into account the condition of Eq. (10), we deduce

$$0=\int_C(-pn_1+vn_2)p^*\mathrm{d}s+\int_D(p\partial p^*/\partial x-v\partial p^*/\partial y)\mathrm{d}a \tag{13}$$

Multiplying the second part of Eq. (8) by v^* and proceeding similarly, we obtain

$$0=\int_C(vn_1+pn_2)v^*\mathrm{d}s-\int_D(v\partial v^*/\partial x+p\partial v^*/\partial y)\mathrm{d}a \tag{14}$$

Summing up these relations, taking into account Eq. (1) and the known formula (see Ref. 6)

$$\int_D p(x-y)\delta(x-\xi,y-\eta)\mathrm{d}x\mathrm{d}y=p(\xi,\eta) \tag{15}$$

we obtain the following integral equation valid for $Q(\xi,\eta)$ in D

$$p(\xi,\eta)+\int_C[(-p^*n_1+v^*n_2)p+(p^*n_2+v^*n_1)v_1]\mathrm{d}s=0 \tag{16}$$

In order to obtain the integral equation on C, we perform $Q(\xi,\eta)\to Q_0(x_0,y_0)\epsilon\ C$. The integral in Eq. (16) becomes singular because of the solution of Eq. (12). A standard calculation yields

$$\frac{1}{2}p(x_0,y_0)+\int_C'[(-p^*n_1+v^*n_2)p+(p^*n_2+v^*n_1)v]\mathrm{d}s=0 \tag{17}$$

Eliminating v from here with the aid of the condition of Eq. (10) we obtain the following integral equation

$$\frac{1}{2}p(x_0,y_0)+\int_C v^*(x-x_0)n_2(x)p(x)\mathrm{d}s$$

$$=\beta\int_C v^*(x-x_0)(n_1^2/n_2)(x)\mathrm{d}s \tag{18}$$

This is the fundamental equation. It determines $p|_C$

IV. Discretization of Equation (18)

The solution of Eq. (18) is carried out by using a collocation method. To this aim the boundary C is divided into N straight – line segments L_j, and it is assumed that on each segment, the function p is constant and equal to its values at the middle point (x_j,y_j) of the segment. Then imposing the condition that Eq. (18) be satisfied at points $(x_0,y_0)=(x_i,y_i)$, the following system is obtained.

$$\frac{1}{2}p_i+\sum_{j=1}^{N}A_{ij}p_j=\beta B_i,\qquad i=i,\ldots,N \tag{19}$$

where $p_i=p(x_i,y_i)$, $p_j=p(x_j,y_j)$

$$A_{ij}=a_{ij}n_2^{(j)},\qquad B_i=\sum_{j=1}^{N}a_{ij}n_1^{(j)2}/n_2^{(j)}$$

$$a_{ij}=\int_{L_j}v^*(x-x_i)\,\mathrm{d}s \tag{20}$$

In order to determine a_{ij}, we shall parametrize the segment L_j. To this aim we shall denote by (x_1,y_1) the coordinates of the first end of the segment L_j and (x_2,y_2) the coordinates of the second end (the sense on L_j being that one leaving the fluid at the left). The coordinates of the current point on L_j read

$$x=\frac{x_1+x_2}{2}+\frac{x_2-x_1}{2}t,\qquad y=\frac{y_1+y_2}{2}+\frac{y_2-y_1}{2}t$$

$$-1\le t\le +1 \tag{21}$$

for $\mathrm{d}s$ we shall have $2\mathrm{d}s=r\mathrm{d}t$ where

$$r=\left[(x_2-x_1)^2+(y_2-y_1)^2\right]^{1/2}$$

and for $n^{(j)}$

$$n^{(j)}=\left(\frac{y_2-y_1}{r},-\frac{x_2-x_1}{r}\right) \tag{22}$$

Taking into account Eq. (12), with notation

$$b=(x_2-x_1)(x_1+x_2-2x_i)+(y_2-y_1)(y_1+y_2-2y_i)$$

$$c=(x_1+x_2-2x_i)^2+(y_1+y_2-2y_i)^2$$

$$d=|(x_2-x_1)(y_1+y_2-2y_i)-(y_2-y_1)(x_1+x_2-2x_i)| \tag{23}$$

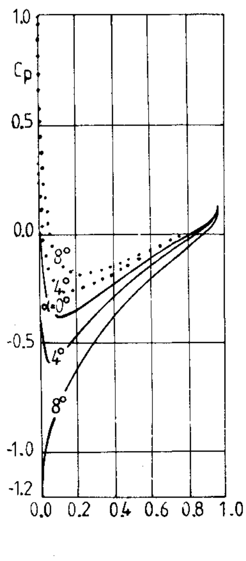

Fig. 1 C_p calculated for an airfoil in incompressible flow.

Table 1 Pressure's value on the circular obstacle

Elements	1	2	3	4	5
Exact	0.987	0.891	0.707	0.454	0.156
This meth.	0.917	0.903	0.717	0.463	0.170
Elements	6	7	8	9	10
Exact	−0.156	−0.454	−0.707	−0.891	−0.987
This meth.	−0.170	−0.463	−0.717	−0.903	−0.917

we deduce

$$2\pi a_{ij} = -r(y_I + y_2 - 2y_1)i_0 - r(y_2 - y_i)i_1 \tag{24}$$

where

$$i_0 = \frac{1}{d}\ \text{arc}\ tg\ \frac{2dr^2}{b^2+d^2-r^4}$$

$$i_1 = \frac{1}{2r^2}\ell n\ \frac{r^2+2b+c}{r^2-2b+c} - \frac{b}{dr^2}\ \text{arc}\ tg\ \frac{2dr^2}{b^2+d^2-r^4} \tag{25}$$

Since $x_1 + x_2 = 2x_j$, $y_1 + y_2 = 2y_j$ we deduce $a_{ii} = 0$.

V. Numerical Results

As a test we use the circular obstacle in an incompressible fluid. In this case the exact solution is known. Table 1 gives the exact values and the values calculated with this method for the pressure on the quarter of the circle starting with the upper point and proceeding clockwise. The pressure distribution is symmetric. The results are very good. This is also confirmed in the case of the elliptic obstable.

In Fig. 1 we gave C_p calculated with the formula

$$C_p = \frac{P_1 - p_\infty}{(1/2)\rho_\infty U_\infty^2} = \frac{2p}{\beta}$$

for an airfoil of aerodynamic interest in incompressible flow at indicated incidences. The data in the catalogues of airfoil sections also confirm these results.

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Analytic Relation for the Holstein-Bohlen Pressure Gradient Parameter

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Introduction

THE classic integral boundary-layer method of Holstein and Bohlen is presented in many texts such as White.[1] The method, while unsatisfactory in some respects, still has academic interest. It was, of course, conceived in an era where hand calculations were necessary. Therefore, the fact that the expressions for the boundary-layer quantities of interest are expressed in terms of the pressure-gradient parameter, Λ for the von Karman-Pohlhausen method instead of the pressure gradient parameter K for the Holstein-Bohlen method, presents no significant difficulty. In order to avoid table look-up operations in computer codes, a function $\Lambda(K)$ would allow routine calculations for all quantities. The analysis reported here resulted in a relatively simple, accurate correlation for $\Lambda(K)$.

In a classic solution of the von Karman[2] momentum integral equation for incompressible flow,

$$\tau_w = \rho_e \frac{\mathrm{d}(u_e{}^2\Theta)}{\mathrm{d}x} + \rho_e u_e \delta^* \frac{\mathrm{d}u_e}{\mathrm{d}x}$$

$$= \rho_e u_e{}^2 \frac{\mathrm{d}\Theta}{\mathrm{d}x} + (2\Theta + \delta^*)\rho_e u_e \frac{\mathrm{d}u_e}{\mathrm{d}x} \tag{1}$$

Pohlhausen[3] assumed the fourth-order polynomial in Eq. (2) for the velocity profile.

$$\frac{u}{u_e} = \sum_{n=0}^{4} a_n(x)\xi^n \qquad \text{for } 0 \le \xi = \frac{y}{\delta(x)} \le 1$$

$$= 1 \qquad \text{for } \xi \ge 1 \tag{2}$$

This polynomial has five coefficients that are related to local conditions on the velocity profile. The minimum conditions are the no-slip boundary condition and the edge condition on velocity. Two conditions guaranteed the asymptotic behavior of the first two derivatives of the velocity at the boundary-layer edge and the last condition was the lowest-order compatibility condition. The resulting velocity profile is

$$\frac{u}{u_e} = 2\xi - 2\xi^3 + \xi^4 + \frac{\Lambda(x)}{6}(\xi - 3\xi^2 + 3\xi^3 - \xi^4)$$

$$\text{for } 0 \le \xi \le 1 \text{ with } \Lambda(x) = \frac{\delta^2}{\upsilon}\frac{\mathrm{d}u_e}{\mathrm{d}x} \tag{3}$$

where Λ is the pressure-gradient parameter. Then, the displacement and momentum thicknesses are related to the boundary-layer thickness by

$$\delta^* = \text{the displacement thickness} = \int_0^\delta \left(1 - \frac{u}{u_e}\right)\mathrm{d}y$$

$$= \frac{\delta}{10}\left(3 - \frac{\Lambda}{12}\right) \tag{4}$$

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